$$\hat{\sigma}_p = \left(\frac{\sum_{i=1}^{N} |x_i - \mu|^p}{N} \right)^{1/p} \quad (10)$$

4.2 Estimation of the shape parameter p

The methods presented in literature are based on the likelihood function and on indices of kurtosis

4.2.1 Estimation of p by means of the maximum likelihood method

If we want to determine the maximum likelihood estimator of the shape parameter p, the equation that we obtain by deriving the log-likelihood function (7) is:

$$\frac{\partial L}{\partial p} = -\frac{N}{p^2}[\log(p) + \Psi(1+1/p) - 1] + \frac{1}{p^2 \sigma_p^2} \sum_{i=1}^{N} |x_i - \mu|^p + \frac{1}{p\sigma_p^p}[\log(\sigma_p) \sum_{i=1}^{N} |x_i - \mu|^p - \sum_{i=1}^{N} |x_i - \mu|^p \log|x_i - \mu|] = 0 \quad (11)$$

where $\Psi(.)$ is the digamma function, that is the first derivative of the logarithm of the gamma function. The equation (11) can be solved by using numerical methods. Moreover, Agr`o [3] uses this method showing that it does not work well for small samples, even though it provides good results for samples of size greater than 50 − 100.

4.2.2 Estimation of p by means of indices of kurtosis

These estimation procedures take into account the relationship between the shape parameter p and the kurtosis. The usually used indices of kurtosis are:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\Gamma(1/p)\Gamma(5/p)}{[\Gamma(3/p)]^2} \quad (12)$$

$$VI = \frac{\sqrt{\mu_2}}{\mu_1} = \frac{\sqrt{\Gamma(1/p)\Gamma(3/p)}}{\Gamma(2/p)} \quad (13)$$

$$I = \frac{1}{VI} \quad (14)$$

$$\beta_p = \frac{\mu_{2p}}{\mu_p^2} = p+1 \quad (15)$$

where

$$\mu_r = \sigma_p^p \, p^{r/p} \frac{\Gamma[(r+1)/p]}{\Gamma(1/p)} \quad (16)$$

is the absolute moment of grade r. The index β_p, called generalized index of kurtosis, the estimators of the indices of kurtosis above described are given by:

$$\hat{\beta}_2 = \frac{n\sum_{i=1}^{n}(x_i - M)^4}{[\sum_{i=1}^{n}(x_i - M)^2]^2} \quad (17)$$

$$\hat{VI} = \frac{\sqrt{n\sum_{i=1}^{n}(x_i - M)^2}}{\sum_{i=1}^{n}|x_i - M|} \quad (18)$$

$$\hat{I} = \frac{1}{\hat{VI}} \quad (19)$$

$$\hat{\beta}_p = \frac{n\sum_{i=1}^{n}|x_i - M|^{2\hat{p}}}{(\sum_{i=1}^{n}|x_i - M|^{\hat{p}})^2} = \hat{p}+1 \quad (20)$$

where M is the arithmetic mean. The following approximation of the inverse of the expression (12) which is obtained by applying the least squared method (LSM) on a generic second-order monotonic analytical expression of (12):

$$\hat{p} \approx \sqrt{\frac{5}{\hat{\beta}_2 - 1.865}} - 0.12 \quad (21)$$

We used alternative method to maximize the ML equation in (7) to ensure the estimated parameters, this done by resorting to the Nelder-Mead (NM) method direct search method. The appeal of the NM optimization technique lies in the fact that it can minimize the negative of the log-likelihood objective function given in (7), essentially without relying on any derivative information. Despite the danger of unreliable performance (especially in high dimensions), numerical experiments have shown that the NM method can converge to an acceptably accurate solution with substantially fewer function evaluations. Good numerical performance and a significant improvement in computational complexity for our estimation method are also insured by obtaining initial estimates from the method of moments. Therefore, optimizations with the NM technique produce a good estimation for three parameters in GLD.